

Boundary value problem with fractional p-Laplacian operator

César Torres

Departamento de Matemáticas
Universidad Nacional de Trujillo
Av. Juan Pablo Segundo s/n, Trujillo-Perú
(ctl576@yahoo.es, ctorres@dim.uchile.cl)

Abstract

The aim of this paper is to obtain the existence of solution for the fractional p-Laplacian Dirichlet problem with mixed derivatives

$${}_tD_T^\alpha (|{}_0D_t^\alpha u(t)|^{p-2} {}_0D_t^\alpha u(t)) = f(t, u(t)), \quad t \in [0, T], \\ u(0) = u(T) = 0,$$

where $\frac{1}{p} < \alpha < 1$, $1 < p < \infty$ and $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function which satisfies some growth conditions. We obtain the existence of nontrivial solution by using the Mountain Pass Theorem.

Key words: Fractional calculus, mixed fractional derivatives, boundary value problem, p-Laplacian operator, mountain pass theorem

MSC

1 Introduction

Recently, a great attention has been focused on the study of boundary value problems (BVP) for fractional differential equations. They appear in mathematical models in different branches in Science as physics, chemistry, biology, geology, as well as, control theory, signal theory, nanoscience and so on [2, 9, 14, 16, 17, 24] and references therein.

Physical models containing left and right fractional differential operators have recently renewed attention from scientists which is mainly due to applications as models for physical phenomena exhibiting anomalous diffusion. Specifically, the models involving a fractional differential oscillator equation, which contains a composition of left and right fractional derivatives, are proposed for the description of the processes of emptying the silo [10] and the heat flow through a bulkhead filled with granular material [19], respectively.

Their studies show that the proposed models based on fractional calculus are efficient and describe well the processes.

The existence and multiplicity of solutions for BVP for nonlinear fractional differential equations is extensively studied using various tools of nonlinear analysis as fixed point theorems, degree theory and the method of upper and lower solutions [3, 4]. Very recently, it should be noted that critical point theory and variational methods have also turned out to be very effective tools in determining the existence of solutions of BVP for fractional differential equations. The idea behind them is trying to find solutions of a given boundary value problem by looking for critical points of a suitable energy functional defined on an appropriate function space. In the last 30 years, the critical point theory has become a wonderful tool in studying the existence of solutions to differential equations with variational structures, we refer the reader to the books due to Mawhin and Willem [11], Rabinowitz [15], Schechter [18] and papers [6, 7, 8, 20, 21, 22, 23, 24].

The p-Laplacian operator was considered in several recent works. It arises in the modelling of different physical and natural phenomena; non-Newtonian mechanics, nonlinear elasticity and glaciology, combustion theory, population biology, nonlinear flow laws, system of Monge-Kantorovich partial differential equations. There exists a very large number of papers devoted to the existence of solutions of the p-Laplacian operator in which the authors used bifurcation, variational methods, sub-super solutions, degree theory, in order to prove the existence of solutions of this nonlinear operator, for detail see [5].

Motivated by these previous works, we consider the solvability of the Dirichlet problem with mixed fractional derivatives

$$\begin{aligned} {}_t D_T^\alpha (|{}_0 D_t^\alpha u(t)|^{p-2} {}_0 D_t^\alpha u(t)) &= f(t, u(t)), \quad t \in [0, T], \\ u(0) &= u(T) = 0, \end{aligned} \quad (1.1)$$

where $1 < p < \infty$, $\frac{1}{p} < \alpha < 1$ and we assume that $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfying:

$\boxed{f_1}$ There exists $C > 0$ and $1 < q < \infty$, such that

$$|f(t, \xi)| \leq C(1 + |\xi|^{q-1}) \quad \text{such that for a.e. } t \in [0, T], \quad \xi \in \mathbb{R}$$

$\boxed{f_2}$ There exists $\mu > p$ and $r > 0$ such that for a.e. $t \in [0, T]$ and $r \in \mathbb{R}$, $|\xi| \geq r$

$$0 < \mu F(t, \xi) \leq \xi f(t, \xi),$$

where $F(t, \xi) = \int_0^\xi f(t, \sigma) d\sigma$.

$$\boxed{f_3} \quad \lim_{\xi \rightarrow 0} \frac{f(t, \xi)}{|\xi|^{p-1}} = 0 \text{ uniformly for a.e. } t \in [0, T].$$

We say that $u \in E_0^{\alpha, p}$ is a weak solution of problem (1.1), if

$$\int_0^T |{}_0D_t^\alpha u(t)|^{p-2} {}_0D_t^\alpha u(t) {}_0D_t^\alpha \varphi(t) dt = \int_0^T f(t, u(t)) \varphi(t) dt,$$

for any $\varphi \in E_0^{\alpha, p}$, where space $E_0^{\alpha, p}$ will be introduced in Section § 2.

Let $I : E_0^{\alpha, p} \rightarrow \mathbb{R}$ the functional associated to (1.1), defined by

$$I(u) = \frac{1}{p} \int_0^T |{}_0D_t^\alpha u(t)|^p dt - \int_{\mathbb{R}} F(t, u(t)) dt \quad (1.2)$$

under our assumption $I \in C^1$ and we have

$$I'(u)v = \int_0^T |{}_0D_t^\alpha u(t)|^{p-2} {}_0D_t^\alpha u(t) {}_0D_t^\alpha v(t) dt - \int_0^T f(t, u(t)) v(t) dt. \quad (1.3)$$

Moreover critical points of I are weak solutions of problem (1.1).

Using the Mountain pass Theorem, we get our main result.

Theorem 1.1. *Suppose that f satisfies $(f_1) - (f_3)$. If $p < q < \infty$ then the problem (1.1) has a nontrivial weak solution in $E_0^{\alpha, p}$.*

The rest of the paper is organized as follows: In Section §2 we present preliminaries on fractional calculus and we introduce the functional setting of the problem. In Section §3 we prove Theorem 1.1.

2 Fractional Calculus

In this section we introduce some basic definitions of fractional calculus which are used further in this paper. For the proof see [9, 14, 17].

Let u be a function defined on $[a, b]$. The left (right) Riemann-Liouville fractional integral of order $\alpha > 0$ for function u is defined by

$$\begin{aligned} {}_aI_t^\alpha u(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \quad t \in [a, b], \\ {}_tI_b^\alpha u(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s) ds, \quad t \in [a, b], \end{aligned}$$

provided in both cases that the right-hand side is pointwise defined on $[a, b]$.

The left and right Riemann - Liouville fractional derivatives of order $\alpha > 0$ for function u denoted by ${}_aD_t^\alpha u(t)$ and ${}_tD_b^\alpha u(t)$, respectively, are defined by

$$\begin{aligned} {}_aD_t^\alpha u(t) &= \frac{d^n}{dt^n} {}_aI_t^{n-\alpha} u(t), \\ {}_tD_b^\alpha u(t) &= (-1)^n \frac{d^n}{dt^n} {}_tI_b^{n-\alpha} u(t), \end{aligned}$$

where $t \in [a, b]$, $n - 1 \leq \alpha < n$ and $n \in \mathbb{N}$.

The left and right Caputo fractional derivatives are defined via the above Riemann-Liouville fractional derivatives [9]. In particular, they are defined for the function belonging to the space of absolutely continuous function, namely, If $\alpha \in (n - 1, n)$ and $u \in AC^n[a, b]$, then the left and right Caputo fractional derivative of order α for function u denoted by ${}_a^c D_t^\alpha u(t)$ and ${}_t^c D_b^\alpha u(t)$ respectively, are defined by

$$\begin{aligned} {}_a^c D_t^\alpha u(t) &= {}_a I_t^{n-\alpha} u^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds, \\ {}_t^c D_b^\alpha u(t) &= (-1)^n {}_t I_b^{n-\alpha} u^{(n)}(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} u^{(n)}(s) ds. \end{aligned}$$

The Riemann-Liouville fractional derivative and the Caputo fractional derivative are connected with each other by the following relations

Theorem 2.1. *Let $n \in \mathbb{N}$ and $n - 1 < \alpha < n$. If u is a function defined on $[a, b]$ for which the Caputo fractional derivatives ${}_a^c D_t^\alpha u(t)$ and ${}_t^c D_b^\alpha u(t)$ of order α exists together with the Riemann-Liouville fractional derivatives ${}_a D_t^\alpha u(t)$ and ${}_t D_b^\alpha u(t)$, then*

$$\begin{aligned} {}_a^c D_t^\alpha u(t) &= {}_a D_t^\alpha u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}, \quad t \in [a, b], \\ {}_t^c D_b^\alpha u(t) &= {}_t D_b^\alpha u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-t)^{k-\alpha}, \quad t \in [a, b]. \end{aligned}$$

In particular, when $0 < \alpha < 1$, we have

$${}_a^c D_t^\alpha u(t) = {}_a D_t^\alpha u(t) - \frac{u(a)}{\Gamma(1-\alpha)} (t-a)^{-\alpha}, \quad t \in [a, b] \quad (2.1)$$

and

$${}_t^c D_b^\alpha u(t) = {}_t D_b^\alpha u(t) - \frac{u(b)}{\Gamma(1-\alpha)} (b-t)^{-\alpha}, \quad t \in [a, b]. \quad (2.2)$$

Now we consider some properties of the Riemann-Liouville fractional integral and derivative operators.

(1)

$$\begin{aligned} {}_a I_t^\alpha ({}_a I_t^\beta u(t)) &= {}_a I_t^{\alpha+\beta} u(t) \quad \text{and} \\ {}_t I_b^\alpha ({}_t I_b^\beta u(t)) &= {}_t I_b^{\alpha+\beta} u(t) \quad \forall \alpha, \beta > 0, \end{aligned}$$

(2) **Left inverse.** Let $u \in L^1[a, b]$ and $\alpha > 0$,

$$\begin{aligned} {}_a D_t^\alpha ({}_a I_t^\alpha u(t)) &= u(t), \text{ a.e. } t \in [a, b] \text{ and} \\ {}_t D_b^\alpha ({}_t I_b^\alpha u(t)) &= u(t), \text{ a.e. } t \in [a, b]. \end{aligned}$$

(3) For $n - 1 \leq \alpha < n$, if the left and right Riemann-Liouville fractional derivatives ${}_a D_t^\alpha u(t)$ and ${}_t D_b^\alpha u(t)$, of the function u are integral on $[a, b]$, then

$$\begin{aligned} {}_a I_t^\alpha ({}_a D_t^\alpha u(t)) &= u(t) - \sum_{k=1}^n [{}_a I_t^{k-\alpha} u(t)]_{t=a} \frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k+1)}, \\ {}_t I_b^\alpha ({}_t D_b^\alpha u(t)) &= u(t) - \sum_{k=1}^n [{}_t I_b^{k-\alpha} u(t)]_{t=b} \frac{(-1)^{n-k} (b-t)^{\alpha-k}}{\Gamma(\alpha-k+1)}, \end{aligned}$$

for $t \in [a, b]$.

(4) **Integration by parts**

$$\int_a^b [{}_a I_t^\alpha u(t)] v(t) dt = \int_a^b u(t) {}_t I_b^\alpha v(t) dt, \quad \alpha > 0, \quad (2.3)$$

provided that $u \in L^p[a, b]$, $v \in L^q[a, b]$ and

$$p \geq 1, \quad q \geq 1 \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \alpha \text{ or } p \neq 1, \quad q \neq 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1 + \alpha.$$

$$\int_a^b [{}_a D_t^\alpha u(t)] v(t) dt = \int_a^b u(t) {}_t D_b^\alpha v(t) dt, \quad 0 < \alpha \leq 1, \quad (2.4)$$

provided the boundary conditions

$$\begin{aligned} u(a) = u(b) = 0, \quad u' \in L^\infty[a, b], \quad v \in L^1[a, b] \text{ or} \\ v(a) = v(b) = 0, \quad v' \in L^\infty[a, b], \quad u \in L^1[a, b], \end{aligned}$$

are fulfilled.

2.1 Fractional Derivative Space

In order to establish a variational structure for BVP (1.1), it is necessary to construct appropriate function spaces. For this setting we take some results from [7, 8, 24].

Let us recall that for any fixed $t \in [0, T]$ and $1 \leq p < \infty$,

$$\|u\|_{L^p[0, t]} = \left(\int_0^t |u(s)|^p ds \right)^{1/p}, \quad \|u\|_{L^p} = \left(\int_0^T |u(s)|^p ds \right)^{1/p} \text{ and } \|u\|_\infty = \max_{t \in [0, T]} |u(t)|.$$

Definition 2.1. Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative spaces $E_0^{\alpha,p}$ is defined by

$$\begin{aligned} E_0^{\alpha,p} &= \{u \in L^p[0, T] / {}_0D_t^\alpha u \in L^p[0, T] \text{ and } u(0) = u(T) = 0\} \\ &= \overline{C_0^\infty[0, T]}^{\|\cdot\|_{\alpha,p}}. \end{aligned}$$

where $\|\cdot\|_{\alpha,p}$ is defined by

$$\|u\|_{\alpha,p}^p = \int_0^T |u(t)|^p dt + \int_0^T |{}_0D_t^\alpha u(t)|^p dt. \quad (2.5)$$

Remark 2.1. For any $u \in E_0^{\alpha,p}$, nothing the fact that $u(0) = 0$, we have ${}_0^cD_t^\alpha u(t) = {}_0D_t^\alpha u(t)$, $t \in [0, T]$ according to (2.1).

Proposition 2.1. [7] Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.

We recall some properties of the fractional space $E_0^{\alpha,p}$.

Lemma 2.1. [7] Let $0 < \alpha \leq 1$ and $1 \leq p < \infty$. For any $u \in L^p[0, T]$ we have

$$\|{}_0I_\xi^\alpha u\|_{L^p[0,t]} \leq \frac{t^\alpha}{\Gamma(\alpha+1)} \|u\|_{L^p[0,t]}, \text{ for } \xi \in [0, t], \quad t \in [0, T]. \quad (2.6)$$

Proposition 2.2. [8] Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, we have

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0D_t^\alpha u\|_{L^p}. \quad (2.7)$$

If $\alpha > 1/p$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\|_\infty \leq \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|{}_0D_t^\alpha u\|_{L^p}. \quad (2.8)$$

Remark 2.2. Let $1/p < \alpha \leq 1$, if $u \in E_0^{\alpha,p}$, then $u \in L^q[0, T]$ for $q \in [p, +\infty]$. In fact

$$\begin{aligned} \int_0^T |u(t)|^q dt &= \int_0^T |u(t)|^{q-p} |u(t)|^p dt \\ &\leq \|u\|_\infty^{q-p} \|u\|_{L^p}^p. \end{aligned}$$

In particular the embedding $E_0^{\alpha,p} \hookrightarrow L^q[0, T]$ is continuous for all $q \in [p, +\infty]$.

According to (2.7), we can consider in $E_0^{\alpha,p}$ the following norm

$$\|u\|_{\alpha,p} = \|{}_0D_t^\alpha u\|_{L^p}, \quad (2.9)$$

and (2.9) is equivalent to (2.5).

Proposition 2.3. [8] *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$ and $\{u_k\} \rightharpoonup u$ in $E_0^{\alpha,p}$. Then $u_k \rightarrow u$ in $C[0, T]$, i.e.*

$$\|u_k - u\|_\infty \rightarrow 0, \quad k \rightarrow \infty.$$

Now, we are going to prove that $E_0^{\alpha,p}$ is uniformly convex, for this fact we consider the following tools (see [1] for more details).

- (1) **Reverse Hölder Inequality:** Let $0 < p < 1$, so that $p' = \frac{p}{p-1} < 0$. If $u \in L^p(\Omega)$ and

$$0 < \int_\Omega |g(x)|^{p'} dx < \infty,$$

then

$$\int_\Omega |f(x)g(x)| dx \geq \left(\int_\Omega |f(x)|^p dx \right)^{1/p} \left(\int_\Omega |g(x)|^{p'} dx \right)^{1/p'}. \quad (2.10)$$

- (2) **Reverse Minkowski inequality:** Let $0 < p < 1$. If $u, v \in L^p(\Omega)$, the

$$\| |u| + |v| \|_{L^p} \geq \|u\|_p + \|v\|_p \quad (2.11)$$

- (3) Let $z, w \in \mathbb{C}$. If $1 < p \leq 2$ and $p' = \frac{p}{p-1}$, then

$$\left| \frac{z+w}{2} \right|^{p'} + \left| \frac{z-w}{2} \right|^{p'} \leq \left(\frac{1}{2}|z|^p + \frac{1}{2}|w|^p \right)^{1/(p-1)}. \quad (2.12)$$

If $2 \leq p < \infty$, then

$$\left| \frac{z+w}{2} \right|^p + \left| \frac{z-w}{2} \right|^p \leq \frac{1}{2}|z|^p + \frac{1}{2}|w|^p \quad (2.13)$$

Lemma 2.2. $(E_0^{\alpha,p}, \|\cdot\|_{\alpha,p})$ is uniformly convex.

Proof. Let $u, v \in E_0^{\alpha,p}$ satisfy $\|u\|_{\alpha,p} = \|v\|_{\alpha,p} = 1$ and $\|u - v\|_{\alpha,p} \geq \epsilon$, where $\epsilon \in (0, 2)$.

Case $p \geq 2$. By (2.13), we have

$$\begin{aligned}
\left\| \frac{u+v}{2} \right\|_{\alpha,p}^p + \left\| \frac{u-v}{2} \right\|_{\alpha,p}^p &= \int_0^T \left| \frac{{}_0D_t^\alpha u(t) + {}_0D_t^\alpha v(t)}{2} \right|^p dt + \int_0^T \left| \frac{{}_0D_t^\alpha u(t) - {}_0D_t^\alpha v(t)}{2} \right|^p dt \\
&\leq \frac{1}{2} \int_0^t |{}_0D_t^\alpha u(t)|^p dt + \frac{1}{2} \int_0^T |{}_0D_t^\alpha v(t)|^p dt \\
&= \frac{1}{2} \|u\|_{\alpha,p}^p + \frac{1}{2} \|v\|_{\alpha,p}^p = 1.
\end{aligned} \tag{2.14}$$

It follows from (2.14) that

$$\left\| \frac{u+v}{2} \right\|_{\alpha,p}^p \leq 1 - \frac{\epsilon^p}{2^p}.$$

Taking $\delta = \delta(\epsilon)$ such that $1 - (\epsilon/2)^2 = (1 - \delta)^p$, we obtain that

$$\left\| \frac{u+v}{2} \right\|_{\alpha,p} \leq (1 - \delta).$$

Case $1 < p < 2$. First, note that

$$\|u\|_{\alpha,p}^{p'} = \left(\int_0^T \left(|{}_0D_t^\alpha u(t)|^{p'} \right)^{p-1} dt \right)^{\frac{1}{p-1}},$$

where $p' = \frac{p}{p-1}$. Using the reverse Minkowski inequality (2.11) and the inequality (2.12), we get

$$\begin{aligned}
&\left\| \frac{u+v}{2} \right\|_{\alpha,p}^{p'} + \left\| \frac{u-v}{2} \right\|_{\alpha,p}^{p'} \\
&= \left[\int_0^T \left(\left| \frac{{}_0D_t^\alpha u(t) + {}_0D_t^\alpha v(t)}{2} \right|^{p'} \right)^{p-1} dt \right]^{\frac{1}{p-1}} + \left[\int_0^T \left(\left| \frac{{}_0D_t^\alpha u(t) - {}_0D_t^\alpha v(t)}{2} \right|^{p'} \right)^{p-1} dt \right]^{\frac{1}{p-1}} \\
&\leq \left[\int_0^T \left(\left| \frac{{}_0D_t^\alpha u(t) + {}_0D_t^\alpha v(t)}{2} \right|^{p'} + \left| \frac{{}_0D_t^\alpha u(t) - {}_0D_t^\alpha v(t)}{2} \right|^{p'} \right)^{p-1} dt \right]^{\frac{1}{p-1}} \\
&\leq \left[\int_0^T \left(\frac{|{}_0D_t^\alpha u(t)|^p}{2} + \frac{|{}_0D_t^\alpha v(t)|^p}{2} \right) dt \right]^{p'-1} \\
&= \left(\frac{1}{2} \|u\|_{\alpha,p}^p + \frac{1}{2} \|v\|_{\alpha,p}^p \right)^{p'-1} = 1.
\end{aligned} \tag{2.15}$$

By (2.15), we have

$$\left\| \frac{u+v}{2} \right\|_{\alpha,p}^{p'} \leq 1 - \frac{\epsilon^{p'}}{2^{p'}}.$$

Taking $\delta = \delta(\epsilon)$ such that $1 - (\epsilon/2)^{p'} = (1 - \delta)^{p'}$, we get the desired claim. \square

3 Proof of Theorem 1.1

Through this section we consider: $p < q$ and $\frac{1}{p} < \alpha \leq 1$. For $u \in E_0^{\alpha,p}$ we define

$$J(u) = \frac{1}{p} \int_0^T |{}_0D_t^\alpha u(t)|^p dt, \quad H(u) = \int_0^T F(t, u(t)) dt,$$

and

$$I(u) = J(u) - H(u).$$

Obviously, the energy functional $I : E_0^{\alpha,p} \rightarrow \mathbb{R}$ associated with problem (1.1) is well defined.

Lemma 3.1. *If f satisfies assumption (f_1) , then the functional $H \in C^1(E_0^{\alpha,p}, \mathbb{R})$ and*

$$\langle H'(u), v \rangle = \int_0^T f(t, u(t))v(t) dt \quad \text{for all } u, v \in E_0^{\alpha,p}.$$

Proof.

(i) H is Gâteaux-differentiable in $E_0^{\alpha,p}$.

Let $u, v \in E_0^{\alpha,p}$. For each $t \in [0, T]$ and $0 < |\sigma| < 1$, by the mean value theorem, there exists $0 < \delta < 1$,

$$\begin{aligned} \frac{1}{\sigma}(F(t, u + \sigma v) - F(t, u)) &= \frac{1}{\sigma} \int_0^{u+\sigma v} f(t, s) ds - \frac{1}{\sigma} \int_0^u f(t, s) ds \\ &= \frac{1}{\sigma} \int_u^{u+\sigma v} f(t, s) ds = f(t, u + \delta \sigma v)v. \end{aligned}$$

By (f_1) and Young's inequality, we get

$$\begin{aligned} |f(t, u + \delta \sigma v)v| &\leq C(|v| + |u + \delta \sigma v|^{q-1}|v|) \\ &\leq C(2|v|^q + |u + \delta \sigma v|^q + 1) \\ &\leq a2^q(|v|^q + |u|^q + 1). \end{aligned}$$

Since $q > 1$, by (2.7) we have $u, v \in L^q[0, T]$. Moreover, the Lebesgue Dominated Convergence Theorem implies

$$\begin{aligned} \lim_{\sigma \rightarrow 0} \frac{1}{\sigma} (H(u + \sigma v) - H(u)) &= \lim_{\sigma \rightarrow 0} \int_0^T f(t, u + \delta \sigma v) v dt \\ &= \int_0^T \lim_{\sigma \rightarrow 0} f(t, u + \delta \sigma v) v dt = \int_0^T f(t, u) v dt. \end{aligned}$$

(ii) Continuity of Gâteaux-derivative.

Let $\{u_n\}, u \in E_0^{\alpha, p}$ such that $u_n \rightarrow u$ strongly in $E_0^{\alpha, p}$ as $n \rightarrow \infty$. Without loss of generality, we assume that $u_n(t) \rightarrow u(t)$ a.e. in $[0, T]$. By (f_1) , for any $I \subset [0, T]$,

$$\begin{aligned} \int_I |f(t, u_n)|^{q'} dt &\leq C^{q'} \int_I (1 + |u_n|^{q-1})^{q'} dt \\ &\leq C^{q'} 2^{q'} \int_I (1 + |u_n|^q) dt \\ &\leq \overline{C} [\mu(I) + \|u_n\|_\infty^q \mu(I)], \end{aligned} \quad (3.1)$$

where μ denotes the Lebesgue measure of I . It follows from (3.1) that the sequence $\{|f(t, u_n) - f(t, u)|^{q'}\}$ is uniformly bounded and equi-integrable in $L^1[0, T]$. The Vitali Convergence Theorem implies

$$\lim_{n \rightarrow \infty} \int_0^T |f(t, u_n) - f(t, u)|^{q'} dt = 0.$$

Thus, by Hölder inequality and Remark 2.2, we obtain

$$\begin{aligned} \|H'(u_n) - H(u)\|_{(E_0^{\alpha, p})^*} &= \sup_{v \in E_0^{\alpha, p}, \|v\|_{\alpha, p} = 1} \left| \int_0^T (f(t, u_n) - f(t, u)) v dt \right| \\ &\leq \|f(t, u_n) - f(t, u)\|_{L^{q'}} \|v\|_{L^q} \\ &\leq K \|f(t, u_n) - f(t, u)\|_{L^{q'}} \\ &\rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$. Hence, we complete the proof of Lemma. \square

Lemma 3.2. *The functional $J \in C^1(E_0^{\alpha, p}, \mathbb{R})$ and*

$$\langle J'(u), v \rangle = \int_0^T |{}_0D_t^\alpha u(t)|^{p-2} {}_0D_t^\alpha u(t) {}_0D_t^\alpha v(t) dt,$$

for all $u, v \in E_0^{\alpha, p}$. Moreover, for each $u \in E_0^{\alpha, p}$, $J'(u) \in (E_0^{\alpha, p})^*$, where $(E_0^{\alpha, p})^*$ denotes the dual of $E_0^{\alpha, p}$.

Proof.

First, it is easy to see that

$$\langle J'(u), v \rangle = \int_0^T |{}_0D_t^\alpha u(t)|^{p-2} {}_0D_t^\alpha u(t) {}_0D_t^\alpha v(t) dt, \quad (3.2)$$

for all $u, v \in E_0^{\alpha,p}$. It follows from (3.2) that for each $u \in E_0^{\alpha,p}$, $J'(u) \in (E_0^{\alpha,p})^*$.

Next, we prove that $J \in C^1(E_0^{\alpha,p}, \mathbb{R})$. For the proof we need the following inequalities, (see [5])

(i) If $p \in [2, \infty)$ then it holds

$$||z|^{p-2}z - |y|^{p-2}y| \leq \beta |z - y| (|z| + |y|)^{p-2} \quad \text{for all } y, z \in \mathbb{R}, \quad (3.3)$$

with β independent of y and z ;

(ii) If $p \in (1, 2]$ then it holds:

$$||z|^{p-2}z - |y|^{p-2}y| \leq \beta |z - y|^{p-1} \quad \text{for all } y, z \in \mathbb{R}, \quad (3.4)$$

with β independent of y and z .

We define $g : E_0^{\alpha,p} \rightarrow L^{p'}[0, T]$ by

$$g(u) = |{}_0D_t^\alpha u|^{p-2} {}_0D_t^\alpha u,$$

for $u \in E_0^{\alpha,p}$. Let us prove that g is continuous.

Case $p \in (2, \infty)$. For $u, v \in E_0^{\alpha,p}$, by (3.3) and Hölder inequality we have:

$$\begin{aligned} \int_0^T |g(u) - g(v)|^{p'} dt &= \int_0^T ||{}_0D_t^\alpha u|^{p-2} {}_0D_t^\alpha u - |{}_0D_t^\alpha v|^{p-2} {}_0D_t^\alpha v| dt \\ &\leq \beta \int_0^T |{}_0D_t^\alpha u - {}_0D_t^\alpha v|^{p'} (|{}_0D_t^\alpha u| + |{}_0D_t^\alpha v|)^{p'(p-2)} dt \\ &\leq \beta \left(\int_0^T |{}_0D_t^\alpha u - {}_0D_t^\alpha v|^p dt \right)^{p'/p} \left(\int_0^T [|{}_0D_t^\alpha u| + |{}_0D_t^\alpha v|]^p dt \right)^{\frac{p'(p-2)}{p}} \\ &= \beta \|{}_0D_t^\alpha u - {}_0D_t^\alpha v\|_{L^p}^{p'} \|{}_0D_t^\alpha u + {}_0D_t^\alpha v\|_{L^p}^{p'(p-2)} \\ &\leq \overline{C} \|u - v\|_{\alpha,p}^{p'} (\|u\|_{\alpha,p} + \|v\|_{\alpha,p})^{p'(p-2)} \end{aligned} \quad (3.5)$$

with \overline{C} constant independent of u and v .

Case $p \in (1, 2]$. For $u, v \in E_0^{\alpha, p}$, by (3.4) it follows

$$\begin{aligned} \int_0^T |g(u) - g(v)|^{p'} dt &= \int_0^T \left| |{}_0D_t^\alpha u|^{p-2} {}_0D_t^\alpha u - |{}_0D_t^\alpha v|^{p-2} {}_0D_t^\alpha v \right|^{p'} dt \\ &\leq \beta \int_0^T |{}_0D_t^\alpha u - {}_0D_t^\alpha v|^{p'(p-1)} dt \\ &\leq \overline{C}_1 \|u - v\|_{\alpha, p}^{p-1} \end{aligned} \quad (3.6)$$

with \overline{C}_1 constant independent of u and v . From (3.5) and (3.6) the continuity of g is obvious.

On the other hand, we claim that

$$\|J'(u) - J'(v)\|_{(E_0^{\alpha, p})^*} \leq K \|g(u) - g(v)\|_{L^{p'}} \quad (3.7)$$

with $K > 0$ constant independent of $u, v \in E_0^{\alpha, p}$. Indeed, by the Hölder inequality we have:

$$\begin{aligned} |\langle J'(u) - J'(v), \varphi \rangle| &\leq \int_0^T |g(u) - g(v)| |{}_0D_t^\alpha \varphi| dt \\ &\leq \left(\int_0^T |g(u) - g(v)|^{p'} dt \right)^{\frac{1}{p'}} \left(\int_0^T |{}_0D_t^\alpha \varphi|^p dt \right)^{\frac{1}{p}} \\ &\leq K \|g(u) - g(v)\|_{L^{p'}} \|\varphi\|_{\alpha, p} \end{aligned}$$

for $u, v, \varphi \in E_0^{\alpha, p}$, proving (3.7).

Now, by the continuity of g and (3.7), the conclusion of the Lemma follows in a standard way. \square

Combining Lemma 3.1 and Lemma 3.2, we get that $I \in C^1(E_0^{\alpha, p}, \mathbb{R})$ and

$$\langle I'(u), v \rangle = \int_0^T |{}_0D_t^\alpha u|^{p-2} {}_0D_t^\alpha u {}_0D_t^\alpha v dt - \int_0^T f(t, u) v dt,$$

for all $u, v \in E_0^{\alpha, p}$.

Lemma 3.3. *Suppose that f satisfies $(f_1) - (f_3)$. Then there exist $\rho > 0$ and $\beta > 0$ such that*

$$I(u) \geq \alpha > 0,$$

for any $u \in E_0^{\alpha, p}$ with $\|u\|_{\alpha, p} = \rho$.

Proof. By assumptions (f_1) and (f_3) , for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for any $\xi \in \mathbb{R}$ and a.e. $t \in [0, T]$, we have

$$|f(t, \xi)| \leq p\epsilon |\xi|^{p-1} + qC_\epsilon |\xi|^{q-1}. \quad (3.8)$$

It follows from (3.8) that

$$|F(t, \xi)| \leq \epsilon |\xi|^p + C_\epsilon |\xi|^q. \quad (3.9)$$

Let $u \in E_0^{\alpha, p}$. By (3.9), Proposition 2.2 and Remark 2.2, we obtain

$$\begin{aligned} I(u) &= \frac{1}{p} \int_0^T |{}_0D_t^\alpha u(t)|^p dt - \int_0^T F(t, u(t)) dt \\ &\geq \frac{1}{p} \int_0^T |{}_0D_t^\alpha u(t)|^p dt - \epsilon \int_0^T |u(t)|^p dt - C_\epsilon \int_0^T |u(t)|^q dt \\ &\geq \frac{1}{p} \|u\|_{\alpha, p}^p - \frac{\epsilon T^\alpha}{\Gamma(\alpha + 1)} \|u\|_{\alpha, p}^p + C_\epsilon \mathcal{K} \|u\|_{\alpha, p}^q \end{aligned} \quad (3.10)$$

where

$$\mathcal{K} = \frac{T^{\alpha q + 1 - \frac{q}{p}}}{(\Gamma(\alpha)[(\alpha - 1)q + 1]^{1/q})^{q-p} \Gamma(\alpha + 1)^p}.$$

Choosing $\epsilon = \frac{\Gamma(\alpha+1)}{2pT^\alpha}$, by (3.10), we have

$$I(u) \geq \frac{1}{2p} \|u\|_{\alpha, p}^p - C \|u\|_{\alpha, p}^q \geq \|u\|_{\alpha, p}^p \left(\frac{1}{2p} - C \|u\|_{\alpha, p}^{q-p} \right),$$

where C is a constant only depending on α, p, T . Now, let $\|u\|_{\alpha, p} = \rho > 0$. Since $q > p$, we can choose ρ sufficiently small such that

$$\frac{1}{2p} - C\rho^{q-p} > 0,$$

so that

$$I(u) \geq \rho^p \left(\frac{1}{2p} - C\rho^{q-p} \right) =: \beta > 0.$$

Thus, the Lemma is proved. \square

Lemma 3.4. *Suppose that f satisfies $(f_1) - (f_3)$. Then there exists $e \in C_0^\infty[0, T]$ such that $\|e\|_{\alpha, p} \geq \rho$ and $I(e) < \beta$, where ρ and β are given in Lemma 3.3.*

Proof. From assumption (f_2) it follows that

$$F(t, \xi) \geq r^{-\mu} \min\{F(t, r), F(t, -r)\} |\xi|^\mu \quad (3.11)$$

for all $|\xi| > r$ and a.e. $t \in [0, T]$. Thus, by (3.11) and $F(t, \xi) \leq \max_{|\xi| \leq r} F(t, \xi)$ for all $|\xi| \leq r$, we obtain

$$F(t, \xi) \geq r^{-\mu} \min\{F(t, r), F(t, -r)\} |\xi|^\mu - \max_{|\xi| \leq r} F(t, \xi) - \min\{F(t, r), F(t, -r)\}, \quad (3.12)$$

for any $\xi \in \mathbb{R}$ and a.e. $t \in [0, T]$. Since $C_0^\infty[0, T] \subset E_0^{\alpha, p}$, we can fix $u_0 \in C_0^\infty[0, T]$ such that $\|u_0\|_{\alpha, p} = 1$. Now, let $\sigma \geq 1$, by (3.12), we have

$$\begin{aligned} I(\sigma u_0) &= \frac{\sigma^p}{p} \|u_0\|_{\alpha, p}^p - \int_0^T F(t, \sigma u_0(t)) dt \\ &\leq \frac{\sigma^p}{p} - r^{-\mu} \sigma^\mu \int_0^T \min\{F(t, r), F(t, -r)\} |u_0(t)|^\mu dt \\ &\quad + \int_0^T \max_{|\xi| \leq r} F(t, \xi) + \min\{F(t, r), F(t, -r)\} dt. \end{aligned}$$

From assumption (f_1) and (f_2) , we get that $0 < F(t, \xi) \leq C(|r| + |r|^q)$ for $|\xi| \leq r$ a.e. $t \in [0, T]$. Thus,

$$0 < \min\{F(t, r), F(t, -r)\} < C(|r| + |r|^q) \text{ a.e. } t \in [0, T].$$

Since $\mu > p$, passing to the limit as $t \rightarrow \infty$, we obtain that $I(tu_0) \rightarrow -\infty$. Thus, the assertion follows by taking $e = Tu_0$ with T sufficiently large. \square

Lemma 3.5. *Suppose that f satisfies $(f_1) - (f_3)$. Then the functional I satisfies (PS) condition.*

Proof. For any sequence $\{u_n\} \subset E_0^{\alpha, p}$ such that $I(u_n)$ is bounded and $I'(u_n) \rightarrow 0$ as $n \rightarrow \infty$, there exists $M > 0$ such that

$$|\langle I'(u_n), u_n \rangle| \leq M \|u_n\|_{\alpha, p} \text{ and } |I(u_n)| \leq M.$$

For each $n \in \mathbb{N}$, we denote

$$\Omega_n = \{t \in [0, T] \mid |u_n(t)| \geq r\}, \quad \Omega'_n = [0, T] \setminus \Omega_n.$$

We have

$$\frac{1}{p} \|u_n\|_{\alpha, p}^p - \left(\int_{\Omega_n} F(t, u_n) + \int_{\Omega'_n} F(t, u_n) \right) \leq M. \quad (3.13)$$

We proceed with obtaining estimations independent of n for the integrals in (3.13). Let $n \in \mathbb{N}$ be arbitrary chosen. From assumption (f_1) , we have

$$|F(t, \xi)| \leq 2C(|\xi|^q + 1). \quad (3.14)$$

If $t \in \Omega'_n$, then $|u_n(t)| < r$ and by (3.14), it follows

$$F(t, u_n) \leq 2C(|u_n|^q + 1) \leq 2C(r^q + 1)$$

and hence

$$\int_{\Omega'_n} F(t, u_n) dt \leq 2CTr^q + T = K_1. \quad (3.15)$$

If $t \in \Omega_n$, then $|u_n(t)| \geq r$ and by (f_2) it holds

$$F(t, u_n) \leq \frac{1}{\mu} f(t, u_n(t)) u_n(t)$$

which gives

$$\int_{\Omega_n} F(t, u_n) dt \leq \int_{\Omega_n} \frac{1}{\mu} f(t, u_n(t)) u_n(t) dt = \frac{1}{\mu} \left(\int_0^T f(t, u_n) u_n dt - \int_{\Omega'_n} f(t, u_n) u_n dt \right) \quad (3.16)$$

By (f_1) , we deduce

$$\begin{aligned} \left| \int_{\Omega'_n} f(t, u_n) u_n dt \right| &\leq \int_{\Omega'_n} C(|u_n| + |u_n|^q) dt \\ &\leq CT r + CT r^q = K_2, \end{aligned}$$

which yields

$$-\frac{1}{\mu} \int_{\Omega'_n} f(t, u_n) u_n dt \leq \frac{K_2}{\mu}. \quad (3.17)$$

Finally, by (3.13), (3.15), (3.16) and (3.17) we obtain

$$\begin{aligned} \frac{1}{p} \|u_n\|_{\alpha, p}^p - \frac{1}{\mu} \int_0^T f(t, u_n) u_n dt &\leq M + K_1 + \frac{K_2}{\mu} = K, \\ \frac{1}{p} \|u_n\|_{\alpha, p}^p - \frac{1}{\mu} \langle H'(u_n), u_n \rangle &\leq K \end{aligned} \quad (3.18)$$

On the other hand, since $|\langle I'(u_n), u_n \rangle| \leq M \|u_n\|_{\alpha, p}$ for $n \geq n_0$. Consequently, for all $n \geq n_0$, we have

$$|\|u_n\|_{\alpha, p}^p - \langle H'(u_n), u_n \rangle| \leq M \|u_n\|_{\alpha, p}$$

which gives

$$-\frac{1}{\mu} \|u_n\|_{\alpha, p}^p - \frac{M}{\mu} \|u_n\|_{\alpha, p} \leq -\frac{1}{\mu} \langle H'(u_n), u_n \rangle. \quad (3.19)$$

Now, from (3.18) and (3.19) it results

$$\left(\frac{1}{p} - \frac{1}{\mu} \right) \|u_n\|_{\alpha, p}^p - \frac{M}{\mu} \|u_n\|_{\alpha, p} \leq K$$

and taking into account that $\mu > p$, we conclude that $\{u_n\}$ is bounded. Since $E_0^{\alpha, p}$ is a reflexive Banach space, up to a subsequence, still denoted by $\{u_n\}$ such that $u_n \rightharpoonup u$ in $E_0^{\alpha, p}$. Then $\langle I'(u_n), u_n - u \rangle \rightarrow 0$. Thus, we obtain

$$\begin{aligned} \langle I'(u_n), u_n - u \rangle &= \int_0^T |{}_0D_t^\alpha u_n|^{p-2} {}_0D_t^\alpha u_n ({}_0D_t^\alpha u_n - {}_0D_t^\alpha u) dt - \int_0^T f(t, u_n) (u_n - u) dt \\ &\rightarrow 0 \end{aligned} \quad (3.20)$$

as $n \rightarrow \infty$. Moreover, by Proposition 2.3,

$$u_n \rightarrow u \text{ strongly in } C[0, T]. \quad (3.21)$$

From (3.21), $\{u_n\}$ is bounded in $C[0, T]$, the by assumption (f_1) , we have

$$\begin{aligned} \left| \int_0^T f(t, u_n)(u_n - u)dt \right| &\leq \int_0^T |f(t, u_n)| |u_n - u| dt \\ &\leq C \int_0^T |u_n - u| dt + C \int_0^T |u_n|^{q-1} |u_n - u| dt \\ &\leq CT \|u_n - u\|_\infty + CT \|u_n\|_\infty^{q-1} \|u_n - u\|_\infty. \end{aligned}$$

This combined with (3.21) follows

$$\lim_{n \rightarrow \infty} \int_0^T f(t, u_n)(u_n - u)dt = 0,$$

hence one has

$$\int_0^T |{}_0D_t^\alpha u_n|^{p-2} {}_0D_t^\alpha u_n ({}_0D_t^\alpha u_n - {}_0D_t^\alpha u) dt \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.22)$$

Using the standard inequality given by

$$\begin{aligned} (|z|^{p-2}z - |y|^{p-2}y)(z - y) &\geq C_p |z - y|^p \text{ if } p \geq 2 \\ (|z|^{p-2}z - |y|^{p-2}y)(z - y) &\geq \tilde{C}_p \frac{|z - y|^2}{(|z| + |y|)^{2-p}} \text{ if } 1 < p < 2. \end{aligned}$$

(see [13]). From which we obtain for $p > 2$

$$\begin{aligned} \int_0^T |{}_0D_t^\alpha u_n - {}_0D_t^\alpha u|^p dt &\leq \frac{1}{C_p} \int_0^T [|{}_0D_t^\alpha u_n|^{p-2} {}_0D_t^\alpha u_n - |{}_0D_t^\alpha u|^{p-2} {}_0D_t^\alpha u] ({}_0D_t^\alpha u_n - {}_0D_t^\alpha u) dt \\ &\rightarrow 0, \end{aligned} \quad (3.23)$$

as $n \rightarrow \infty$. For $1 < p < 2$, by reverse Hölder inequality, we have

$$\begin{aligned} \int_0^T |{}_0D_t^\alpha u_n - {}_0D_t^\alpha u|^p dt &\leq \tilde{C}_p^{-\frac{p}{2}} \left(\int_0^T (|{}_0D_t^\alpha u_n| + |{}_0D_t^\alpha u|)^p dt \right)^{\frac{2-p}{2}} \\ &\quad \times \left(\int_0^T [|{}_0D_t^\alpha u_n|^{p-2} {}_0D_t^\alpha u_n - |{}_0D_t^\alpha u|^{p-2} {}_0D_t^\alpha u] ({}_0D_t^\alpha u_n - {}_0D_t^\alpha u) dt \right)^{p/2} \\ &\leq \bar{C} \left(\int_0^T [|{}_0D_t^\alpha u_n|^{p-2} {}_0D_t^\alpha u_n - |{}_0D_t^\alpha u|^{p-2} {}_0D_t^\alpha u] [{}_0D_t^\alpha u_n - {}_0D_t^\alpha u] dt \right)^{p/2} \\ &\rightarrow 0, \end{aligned} \quad (3.24)$$

as $n \rightarrow \infty$. Combining (3.23) with (3.24), we get that $u_n \rightarrow u$ strongly in $E_0^{\alpha,p}$ as $n \rightarrow \infty$. Therefore, I satisfies (PS) condition. \square

Proof of Theorem 1.1. Since Lemma 3.3 - Lemma 3.5 hold, the Mountain pass Theorem (see [15]) gives that there exists a critical point $u \in E_0^{\alpha,p}$ of I . Moreover,

$$I(u) \geq \beta > 0 = I(0).$$

Thus, $u \neq 0$. \square

References

- [1] R. Adams and J. Fournier, “*Sobolev space, second ed.*”, Academic Press, New York-London, 2003.
- [2] D. Baleanu, Z. Güvenc and J. Machado (eds), “*New trends in nanotechnology and fractional calculus applications*”, Singapore 2010.
- [3] M. Belmekki, J. Nieto and R. Rodríguez-López, “*Existence of periodic solution for a nonlinear fractional differential equation*”, Bound. Value Probl. 2009, Art. ID 324561, 18 pp. (2009).
- [4] M. Benchohra, A. Cabada and D. Seba, “*An existence result for nonlinear fractional differential equations on Banach spaces*”. Bound. Value Probl. 2009, Article ID 628916, 11 pp. (2009).
- [5] G. Dinca, P. Jebelean and J. Mawhin, “*Variational and topological methods for Dirichlet problems with p -Laplacian*”, Portugal. Math. (N.S.) 58, 339-378 (2001).
- [6] V. Ervin and J. Roop, “*Variational formulation for the stationary fractional advection dispersion equation*”, Numer. Meth. Part. Diff. Eqs, **22**, 58-76(2006).
- [7] F. Jiao and Y. Zhou, “*Existence of solution for a class of fractional boundary value problems via critical point theory*”. Comp. Math. Appl., **62**, 1181-1199(2011).
- [8] F. Jiao and Y. Zhou, “*Existence results for fractional boundary value problem via critical point theory*”, Intern. Journal of Bif. and Chaos, **22**, N 4, 1-17(2012).
- [9] A. Kilbas, H. Srivastava and J. Trujillo, “*Theory and applications of fractional differential equations*”, North-Holland Mathematics Studies, vol 204, Amsterdam, 2006.

- [10] S. Leszczynski and T. Blaszczyk, “*Modeling the transition between stable and unstable operation while emptying a silo*”, Granular Matter **13**, 429-438 (2011).
- [11] J. Mawhin and M. Willen, “*Critical point theory and Hamiltonian systems*”, Applied Mathematical Sciences 74, Springer, Berlin, 1989.
- [12] D. Motreanu and P. Panagiotopoulos, “*Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities*”, Kluwer Academic Publishers, Dordrecht, 1999.
- [13] I. Peral, “*Multiplicity of solutions for the p -Laplacian*”, Second School of nonlinear functional analysis and applications to differential equations (ICTP, Trieste, 1997).
- [14] I. Podlubny, “*Fractional differential equations*”, Academic Press, New York, 1999.
- [15] P. Rabinowitz, “*Minimax method in critical point theory with applications to differential equations*”, CBMS Amer. Math. Soc., No **65**, 1986.
- [16] J. Sabatier, O. Agrawal and J. Tenreiro Machado, “*Advances in fractional calculus. Theoretical developments and applications in physics and engineering*”, Springer-Verlag, Berlin, 2007.
- [17] S. Samko, A. Kilbas and O. Marichev “*Fractional integrals and derivatives: Theory and applications*”, Gordon and Breach, New York, 1993.
- [18] M. Schechter, “*Linking methods in critical point theory*”, Birkhäuser, Boston, 1999.
- [19] E. Szymanek, “*The application of fractional order differential calculus for the description of temperature profiles in a granular layer*”, in Theory & Appl. of Non - integer Order Syst.. W. Mitkowski et al. (Eds.), LNEE **275**, Springer Inter. Publ. Switzerland, 243-248(2013).
- [20] C. Torres, “*Existence of solution for fractional Hamiltonian systems*”, Electronic Jour. Diff. Eq. **2013**, 259, 1-12(2013).
- [21] C. Torres, “*Mountain pass solution for a fractional boundary value problem*”, Journal of Fractional Calculus and Applications, **5**, 1, 1-10(2014).
- [22] C. Torres, “*Existence of a solution for fractional forced pendulum*”, Journal of Applied Mathematics and Computational Mechanics, **13**, 1, 125-142(2014).

- [23] W. Xie, J. Xiao and Z. Luo, “*Existence of Solutions for Fractional Boundary Value Problem with Nonlinear Derivative Dependence*”, Abstract and applied analysis, Article ID 812910, 8 pages, 2014
- [24] Y. Zhou, “*Basic theory of fractional differential equations*”, World Scientific Publishing Co. Pte. Ltd. 2014